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Kinetics calculation on the shear viscosity in hot QED at finite density

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Abstract. The shear viscosity of QED plasma at finite temperature and density is calculated by solving Boltzmann equation with variational approach. The result shows the small chemical potential enhances the viscosity in leading-log order by adding a chemical potential quadratic term to the viscosity for the pure temperature environment.

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1 Introduction

A novel state of matter, strongly interacting quark–gluon plasma (sQGP) is claimed to be found at the Relativistic Heavy Ion Collider at Brookhaven National Laboratory [1]. The measured v_2 was found to reach the hydrodynamic limit of an almost perfect fluid with very small viscosity at low transverse momentum region. It is desirable to explain this near-perfect fluid behavior of sQGP from the theoretical points of view [2].

In principle, there are two approaches to calculate transport coefficients. One is using the Kubo formulae [3] within thermal field theory. In this frame Jeon first evaluated the viscosity via resumming an infinite series of ladder diagrams in relativistic scalar field [4]. Then the authors of [5] studied the shear viscosity in weakly coupled hot ϕ^4 theory using the closed time path formalism (CTP) and another deriviation in the real-time formalism on a general non-perturbative expression appeared in [6]. In [7] the shear viscosity is also computed in the $O(N)$ model in the large N limit. The alternative framework is the kinetics theory [8–11]. Although the transport equations are hard to solve, the relaxation time approximation (RTA) and variational calculus are two popular methods to obtain the transport coefficients. In RTA, one uses classic kinetic formulae, but involving the relativistic and quantum effects, to estimate the shear viscosity [12–14]. Arnold, Moore and Yaffe [9,10] have studied the leading-log contribution as well as the full leading order contribution of various transport coefficients of the QCD-like theory at high temperature by solving the Boltzmann equation with a variational approach. The results in the two frameworks are coincident in leading-log order except for some factor differences. Some publications also demonstrated that the diagrammatic expansion of the Kubo formula was actually equivalent to the kinetics calculation from the linearized Boltzmann equation if all the possible ladder diagrams were resummed in scalar field [4,15] and in pure gauge QCD theory [16] as well as including quarks [17]. In addition, one should pay attention to the consistency of the Ward identity with the ladder resummation [18] in gauge theory.

However, most works listed above concentrated on the high temperature but vanishing chemical potential except [14]. While actually the net baryon number in the central fire ball of heavy-ion collision is not zero rigidly though small [19]. It makes sense to involve this density effect by introducing a chemical potential μ, which is much smaller than the temperature, to study how it affects the shear viscosity of the plasma.

In this paper, we shall try to solve the Boltzmann equation by the variational method at high temperature with finite density in QED, following the scheme in [9] for high temperature and zero chemical potential. QED is a good toy model for the non-Abelian gauge QCD yet simpler in computation. We found the shear viscous coefficient is proportional to $T^3/e^4 \ln \frac{1}{e}$ and modified by a small factor of $(1 + 0.13\mu^2/T^2)$.

The paper is arranged as follows: in Sect. 2 we will review the sketch of solving Boltzmann equation by variational method in the kinetics of transport theory and define the shear viscosity in this framework. The associated collision processes on the right hand side of Boltzmann equation will be calculated in Sect. 3. In Sect. 4 we use the variational method to obtain the shear viscosity. Section 5 gives our conclusions and an outlook.

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We use the notation $P = (p_0, \mathbf{p})$ and $p \equiv |\mathbf{p}|$. The momentum denoted by a capital letter is the four-component momentum and the lowercase one with bold face denotes the three-component momentum.

2 Boltzmann equation and viscosity

Considering a system which slightly deviates from the equilibrium state by a small velocity gradient, one can describe it with the one particle distribution $f(\mathbf{p}'x, t)$ which satisfies the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v_p}\cdot\frac{\partial}{\partial \mathbf{x}} + \mathbf{F}\cdot\frac{\partial}{\partial \mathbf{p}}\right) f(\mathbf{p};\mathbf{x},t) = -C[f], \qquad (1)$$

where $\mathbf{v_p} = \hat{\mathbf{p}} \equiv \mathbf{p}/p$ and $\mathbf{F}$ is the external force. In the case of shear viscosity, the external field is irrelevant and the time derivation on the left hand side may be dropped due to its higher order contribution in spatial gradients [9]. The right hand side of (1) is the collision term which takes the usual form of

$$\begin{aligned} C[f](\mathbf{p}) = \frac{1}{2}\int_{\mathbf{p}',\mathbf{k},\mathbf{k}'} & |\mathcal{M}(p,k;p',k')|^2 \\ & \times(2\pi)^4\delta^{(4)}(P+K-P'-K') \qquad (2)\\ & \times\{f(\mathbf{p})f(\mathbf{k})[1\pm f(\mathbf{p})][1\pm f(\mathbf{k})] \\ & - f(\mathbf{p}')f(\mathbf{k}')[1\pm f(\mathbf{p}')][1\pm f(\mathbf{k}')]\}, \end{aligned}$$

if only $2 \to 2$ elastic collisions are involved. Here $\mathbf{p}, \mathbf{k}, \mathbf{p}'$ and $\mathbf{k}'$ denote the momenta of the incoming and outgoing particles respectively. The momentum space integration $\int_{\mathbf{p}}$ is a shorthand for

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2p_0},$$

and $|\mathcal{M}|^2$ is the two-body scattering amplitude. The $1 \pm f$ factor is the final state statistical weight for boson with the upper sign and for fermion with the down sign.

Following [9], we expand the distribution function in the near-equilibrium state and obtain the linearized Boltzmann equation:

$$S_{ij}(\mathbf{p}) = \mathcal{C}\chi_{ij}(\mathbf{p}), \qquad (3)$$

where $\mathcal{C}$ is the linearized collision operator. By defining the inner product in function space and varying the trial function $\chi_{ij}(\mathbf{p})$, one can obtain the shear viscous coefficient

$$\eta = \frac{2}{15} Q_{\max}, \qquad (4)$$

where

$$Q_{\max} = \frac{1}{2}(\chi_{ij}, \mathcal{C}\chi_{ij})|_{\chi=\chi_{\max}} = \frac{1}{2}(\chi_{ij}, S_{ij})|_{\chi=\chi_{\max}} (5)$$

$$\chi_{ij}(\mathbf{p}) = I_{ij}(\hat{\mathbf{p}})\chi(p) = \sqrt{\frac{3}{2}}\left(\hat{p}_i\hat{p}_j - \frac{1}{3}\delta_{ij}\right)\chi(p), \qquad (6)$$

$$(\chi_{ij}, S_{ij}) = -\beta^2 \sum_a \int_{\mathbf{p}} \mathbf{p}\; f_0^a(\mathbf{p})[1 \pm f_0^a(\mathbf{p})]\chi^a(p), \qquad (7)$$

and in an explicit form the collision term at the right hand side of the Boltzmann equation is

$$\begin{aligned} &(\chi_{ij}, \mathcal{C}\chi_{ij}) \\ &= \frac{\beta^2}{8}\int_{\mathbf{p},\mathbf{k},\mathbf{k},\mathbf{k}'} \sum_{abcd} |\mathcal{M}^{ab}_{cd}|^2 (2\pi)^4\delta^{(4)}(P+K-P'-K') \\ &\quad \times f_0^a(\mathbf{p})f_0^b(\mathbf{k})[1\pm f_0^c(\mathbf{p}')][1\pm f_0^d(\mathbf{k}')] \\ &\quad \times \left[\chi^a_{ij}(\mathbf{p}) + \chi^b_{ij}(\mathbf{k}) - \chi^c_{ij}(\mathbf{p}') - \chi^d_{ij}(\mathbf{k}')\right]^2, \qquad (8) \end{aligned}$$

where f_0 represents for the local equilibrium distribution function. a, b, c and d are for the species of particles.

In the above definitions, we adopted the formalisms developed by Arnold, Moore and Yaffe [9] with the only differences in the distribution functions which involved the chemical potential in the initial and the final states. Another notation one should notice is the sum in front of the matrix element which means all possible collision processes relevant to the leading-log contribution are involved and properly treated without double counting or multi-counting.

3 Collision terms

In QED, all possible reactions can be classified in two categories: processes of exchanging a boson (Fig. 1a) and processes of exchanging a fermion (Fig. 1b,c), in which the later includes the pair production and the Compton scattering processes. Notice that the s-channel scattering is omitted because it is infrared finite and thus does not contribute to the leading-log result.

Before going into the next step of the calculation, we should specify some important approximations and definitions.
(1) In our discussion, we adopt the hard forward scattering approximation, namely the momentum transfer $q \sim eT$ which is small for all the time since it is sufficient to compute the leading-log viscosity. So we neglect all the differences between the distribution functions such as $n_f(p)$ and $n_f(p')$. In addition, the fermion mass is also omitted in this case, for it is in order of eT which is much smaller than the hard scale T. Thus the kinematics of the two-body collision gives

$$\cos\theta_{pk} = 1 + (1-\cos^2\theta)(1-\cos\phi), \qquad (9)$$

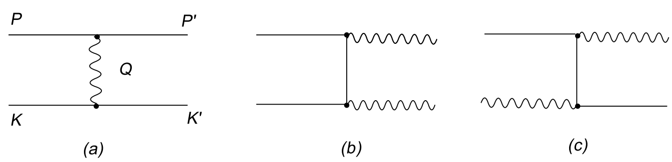

Fig. 1a–c. The possible processes which contribute to the leading-log in the collision term in QED plasma. The solid line is for an electron and the wiggly line is for a photon

where θ_{pk} is the angle between $\mathbf{p}$ and $\mathbf{k}$. θ is the angle between $\mathbf{p}$ and $\mathbf{q}$, and the angle between $\mathbf{k}$ and $\mathbf{q}$ as well, since they are approximately equal in the forward scattering. ϕ is the angle between the $\mathbf{p}$–$\mathbf{q}$ plane and the $\mathbf{p}$–$\mathbf{q}$ plane.
(2) Due to the energy-momentum conservation, only three of the four momenta of incoming and outgoing particles are independent. If we properly label the particles coming from the same vertex with the similar momentum symbols as shown in the Fig. 1a, for example P and P', all the three Mandelstam variables can be defined as $s = (P + K)^2$, $t = (P - P')^2$ and $u = (P - K')^2$.
(3) As to the infrared divergence, the two categories of collisions behave in different ways. When the momentum transfer $q \equiv |\mathbf{p} - \mathbf{p}'|$ goes to zero in the forward scattering, one finds that the infrared singularity in the fermion-exchange process is logarithmical while in the boson-exchange process it is quadratic. Fortunately that is not so bad for the latter case because if carefully considering the $[\chi^a + \chi^b - \chi^c - \chi^d]^2$ term one may find that a small q^2 emerges which softens the quadratic divergence into a logarithmical one. Since now all the collision integrations are logarithmically divergent, the limit cut-offs play important roles in our calculation. For transport coefficients like shear viscosity, these integrations are dominated by the hard scale T of the system which can be chosen as the ultraviolet cut-off. As to the infrared limit, the hard thermal loop self-energy scale eT is sufficient [9]. Even in the finite density case, the small chemical potential only modifies the infrared cut-off by adding a factor like $e\mu$ behind eT, which does not contribute to the leading-log order $\ln 1/e$ since we assume the chemical potential is much smaller than the typical momentum scale T, i.e. $\mu \le eT \ll T$. Therefore we will not carefully treat the $\mathrm{d}q$ integration and just adopt T and eT as the upper and down limits respectively.

Now let us continue our calculation. $\delta^3(\mathbf{p}+\mathbf{k}-\mathbf{p}'-\mathbf{k}')$ in the integrand of (8) helps to perform the $\mathbf{k}'$ integration, yet to the δ function of energy conservation, one may introduce a dummy integration variable ω [8]

$$\delta(p+k-p'-k') = \int_{-\infty}^{\infty} \mathrm{d}\omega \delta(\omega+p-p')\delta(\omega-k+k'). \quad (10)$$

With this trick we can integrate over the angles and the remaining integrals are

$$\begin{aligned}&(\chi_{ij}, \mathcal{C}\chi_{ij})\\ &= \frac{\beta^3}{(4\pi)^6} \int_0^\infty \mathrm{d}q \int_{-q}^{q} \mathrm{d}\omega \int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}k \int_0^{2\pi} \mathrm{d}\phi\\ &\quad \times \sum_{abcd} |\mathcal{M}^{ab}_{cd}|^2 \times f_0^a(\mathbf{p}) f_0^b(\mathbf{k}) [1 \pm f_0^c(\mathbf{p})][1 \pm f_0^d(\mathbf{k})]\\ &\quad \times \left[\chi^a_{ij}(\mathbf{p}) + \chi^b_{ij}(\mathbf{k}) - \chi^c_{ij}(\mathbf{p}') - \chi^d_{ij}(\mathbf{k}')\right]^2, \end{aligned} \quad (11)$$

with $p' = p + \omega$ and $k' = k - \omega$. For the sake of convenience, we adopt $n_f(p)$ as the fermion distribution and $b(p)$ for the boson function in the equilibrium state in the following calculation.

3.1 Boson-exchange processes

Unlike the pure temperature case, the system with finite chemical potential requires a more careful treatment to distinguish the different species of fermions with different distribution functions. For the boson-exchange process, Bhabha scattering $e^+e^- \to e^+e^-$ and Møller scattering $e^-e^- \to e^-e^-$ or $e^+e^+ \to e^+e^+$ have been involved. The s-channel reaction has been omitted since it does not contribute to the leading-log order, and the distribution functions in the Boltzmann equation for both scatterings are

$$\begin{aligned}&\text{Bhabha scattering}:\\ &\quad 2\bar{n}_f(p)\bar{n}_f(k)[1-\bar{n}_f(p)][1-\bar{n}_f(k)]\\ &\qquad +2n_f(p)n_f(k)[1-n_f(p)][1-n_f(k)],\\ &\text{Møller scattering}:\\ &\quad 4\bar{n}_f(p)n_f(k)[1-\bar{n}_f(p)][1-n_f(k)], \end{aligned} \quad (12)$$

where the extra factor of 4 in the Møller scattering process is from the sum over the initial and final states, and the factor of 2 in the Bhabha scattering comes from the t-channel and u-channel contributions respectively. $\bar{n}_f(p) = [\mathrm{e}^{\beta(p+\mu)} + 1]^{-1}$ is the distribution function for the positron and $n_f(p) = [\mathrm{e}^{\beta(p-\mu)} + 1]^{-1}$ is the distribution function for the electron.

In the forward scattering case, one can easily check $s \approx -u$; thus the matrix element for the t-channel is

$$8e^4 \frac{s^2+u^2}{t^2} \approx 16e^4 \frac{u^2}{t^2} = 16e^4 \frac{4p^2k^2}{q^2}(1-\cos\phi)^2, \quad (13)$$

where the spins of initial and final states have been summed. As to the u-channel, the matrix element is identical with that of the t-channel as long as the momentum symbols are well defined.

In the case of small q and the particle species a, b being identical to c, d(or d, c) respectively, one finds

$$\begin{aligned}&\chi^{\mathrm{e}}_{ij}(\mathbf{p}') - \chi^{\mathrm{e}}_{ij}(\mathbf{p}) = \mathbf{q}\cdot\nabla\chi^{\mathrm{e}}_{ij}(\mathbf{p}) + \dots \quad (14)\\ &\approx \omega I_{ij}(\hat{\mathbf{p}})\chi^{\mathrm{e}}(p)' - \sqrt{\frac{3}{2}}(2\omega\hat{p}_i\hat{p}_i - q_i\hat{p}_j - q_j\hat{p}_i)\frac{\chi^{\mathrm{e}}(p)}{p}, \end{aligned}$$

where $\chi^{\mathrm{e}}(p)' = \mathrm{d}\chi^{\mathrm{e}}(p)/\mathrm{d}p$. The square of the above equation is

$$\begin{aligned}&[\chi^{\mathrm{e}}(\mathbf{p}') - \chi^{\mathrm{e}}(\mathbf{p}')]^2 \quad (15)\\ &= \omega^2[\chi^{\mathrm{e}}(p)']^2 + 3\frac{q^2-\omega^2}{p^2}[\chi^{\mathrm{e}}(p)]^2 + \mathcal{O}(q^3). \end{aligned}$$

Here, the electron and positron have the same departure from the equilibrium which is denoted by χ^{e}. One can prove that the cross terms like $[\chi^{\mathrm{e}}_{ij}(\mathbf{p}')-\chi^{\mathrm{e}}_{ij}(\mathbf{p})]\cdot[\chi^{\mathrm{e}}_{ij}(\mathbf{k}')-\chi^{\mathrm{e}}_{ij}(\mathbf{k})]$ vanish when carrying out the $\mathrm{d}\omega$ and $\mathrm{d}\phi$ integration with the factor $(1-\cos\phi)^2$ coming from the matrix element.

Combining (11)–(13) and completing the $\mathrm{d}\omega$ and $\mathrm{d}\phi$ integration, we obtain the collision term for the boson-exchange process:

$$(\chi_{ij}\mathcal{C}\chi_{ij})^{(a)}$$

$$
\begin{aligned}
&= \frac{\beta^3}{(2\pi)^3}\int_{eT}^{T}\frac{\mathrm{d}q}{q}\int_0^\infty \mathrm{d}p\int_0^\infty \mathrm{d}k p^2k^2 \\
&\quad\times\left\{p^2[\chi^{\mathrm{e}}(p)']^2+6[\chi^{\mathrm{e}}(p)]^2\right\} \\
&\quad\times\{n_f(p)n_f(k)[1-n_f(p)][1-n_f(k)] \\
&\quad+\bar{n}_f(p)\bar{n}_f(k)[1-\bar{n}_f(p)][1-\bar{n}_f(k)] \\
&\quad+\bar{n}_f(p)n_f(k)[1-\bar{n}_f(p)][1-n_f(k)] \\
&\quad+n_f(p)\bar{n}_f(k)[1-n_f(p)][1-\bar{n}_f(k)]\}\ , \qquad (16)
\end{aligned}
$$

where we have replaced k with p in the χ-functions and placed an extra factor of 2 in front of the remaining integration.

Noticing that the k integration can be done after expanding the distribution functions in terms of μ/T,

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}kk^2 n_f(k)[1-n_f(k)] \\
&\approx T^3\left[\frac{\pi^2}{6}+\ln 4\frac{\mu}{T}+\frac{\mu^2}{2T^2}\right], \\
&\int_0^\infty \mathrm{d}kk^2 \bar{n}_f(k)[1-\bar{n}_f(k)] \\
&\approx T^3\left[\frac{\pi^2}{6}-\ln 4\frac{\mu}{T}+\frac{\mu^2}{2T^2}\right],
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{(\chi_{ij},\mathcal{C}\chi_{ij})^{(a)}}{e^4\ln\frac{1}{e}} \\
&\approx\left(1+\frac{3}{\pi^2}\frac{\mu^2}{T^2}\right)\int_0^\infty \mathrm{d}p \\
&\quad\times\{n_f(p)[1-n_f(p)]+\bar{n}_f(p)[1-\bar{n}_f(p)]\} \\
&\quad\times\left\{p^2[\chi^{\mathrm{e}}(p)']^2+6[\chi^{\mathrm{e}}(p)]^2\right\}. \qquad (17)
\end{aligned}
$$

3.2 Pair production

The pair production process is described by Fig. 1b and its reversed process. The typical matrix element for this process is

$$
|\mathcal{M}^{ee}_{\gamma\gamma}|^2=\frac{u}{t}+\frac{t}{u}\to\frac{2u}{t}=8e^4\frac{2pk}{q^2}(1-\cos\phi). \qquad (18)
$$

Adding the distribution functions contributions, (11) is recast into

$$
\begin{aligned}
&(\chi_{ij},\mathcal{C}\chi_{ij})^{(b)} \\
&=\frac{16e^4}{(4\pi)^6}\iint_0^\infty \mathrm{d}p\mathrm{d}q\mathrm{d}k\int_{-q}^{q}\mathrm{d}\omega\int_0^{2\pi}\mathrm{d}\phi(1-\cos\phi)\frac{2pk}{q^2} \\
&\quad\times[\chi^a_{ij}(\mathbf{p})+\chi^b_{ij}(\mathbf{k})-\chi^c_{ij}(\mathbf{p}')-\chi^d_{ij}(\mathbf{k}')]^2 \\
&\quad\times\{n_f(p)\bar{n}_f(k)[1+b(p)][1+b(k)] \\
&\quad+\bar{n}_f(p)n_f(k)[1+b(p)][1+b(k)] \\
&\quad+b(p)b(k)[1-\bar{n}_f(p)][1-n_f(k)] \\
&\quad+b(p)b(k)[1-n_f(p)][1-\bar{n}_f(k)]\}. \qquad (19)
\end{aligned}
$$

Expanding the χ-function term and ignoring the momenta difference between the incoming and outgoing particles we get

$$
\begin{aligned}
&[\chi^{\mathrm{e}}_{ij}(\mathbf{p})+\chi^{\mathrm{e}}_{ij}(\mathbf{k})-\chi^{\gamma}_{ij}(\mathbf{p}')-\chi^{\gamma}_{ij}(\mathbf{k}')]^2 \\
&\approx I^2_{ij}(\hat{p})[\chi^{\mathrm{e}}(p)-\chi^{\gamma}(p)]^2 \\
&\quad+I^2_{ij}(\hat{k})[\chi^{\mathrm{e}}(k)-\chi^{\gamma}(k)]^2 \qquad (20)\\
&\quad+2I_{ij}(\hat{p})\cdot I_{ij}(\hat{k})[\chi^{\mathrm{e}}(p)-\chi^{\gamma}(p)][\chi^{\mathrm{e}}(k)-\chi^{\gamma}(k)].
\end{aligned}
$$

Noticing that $I^2_{ij}(\hat{p})=1$ and

$$
I_{ij}(\hat{p})\cdot I_{ij}(\hat{k})=\frac{1}{2}(3\cos^2\theta_{pk}-1)=P_2(\cos\theta_{pk}), \qquad (21)
$$

where $P_2(\cos\theta_{pk})$ is the second Legendre polynomial, one can check that the cross term vanishes when integrating over $\mathrm{d}\phi$. We carry out the k integration by expanding the integrand in terms of small μ/T and find

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}kk\bar{n}_f(k)[1+b(k)] \\
&\approx T^2\left[\frac{\pi^2}{8}-0.963\frac{\mu}{T}+0.298\frac{\mu^2}{T^2}\right], \\
&\int_0^\infty \mathrm{d}kkn_f(k)[1+b(k)] \\
&\approx T^2\left[\frac{\pi^2}{8}+0.963\frac{\mu}{T}+0.298\frac{\mu^2}{T^2}\right], \\
&\int_0^\infty \mathrm{d}kkb(k)[1-n_f(k)] \\
&\approx T^2\left[\frac{\pi^2}{8}-0.270\frac{\mu}{T}-0.048\frac{\mu^2}{T^2}\right], \\
&\int_0^\infty \mathrm{d}kkb(k)[1-\bar{n}_f(k)] \\
&\approx T^2\left[\frac{\pi^2}{8}+0.270\frac{\mu}{T}-0.048\frac{\mu^2}{T^2}\right]. \qquad (22)
\end{aligned}
$$

Then (19) becomes

$$
\begin{aligned}
&(\chi_{ij},\mathcal{C}\chi_{ij})^{(b)} \\
&=\frac{\beta e^4\ln\frac{1}{e}}{2^4\pi^5}\int_0^\infty \mathrm{d}pp[\chi^{\mathrm{e}}(p)-\chi^{\gamma}(p)]^2 \qquad (23)\\
&\quad\times\left\{n_f(p)[1+b(p)]\left(\frac{\pi^2}{8}-0.963\frac{\mu}{T}+0.298\frac{\mu^2}{T^2}\right)\right. \\
&\quad+\bar{n}_f(p)[1+b(p)]\left(\frac{\pi^2}{8}+0.963\frac{\mu}{T}+0.298\frac{\mu^2}{T^2}\right) \\
&\quad+b(p)[1-\bar{n}_f(p)]\left(\frac{\pi^2}{8}-0.27\frac{\mu}{T}-0.048\frac{\mu^2}{T^2}\right) \\
&\quad\left.+b(p)[1-n_f(p)]\left(\frac{\pi^2}{8}+0.27\frac{\mu}{T}-0.048\frac{\mu^2}{T^2}\right)\right\}.
\end{aligned}
$$

3.3 Compton scattering

The Compton scattering process involves both electron and position contributions. The matrix element for this process is

$$|\mathcal{M}_{e\gamma}^{e\gamma}|^2 = -8e^4\frac{s}{u} = 8e^4\frac{2pk}{q^2}(1-\cos\phi). \tag{24}$$

The distribution functions for this process is

$$\begin{aligned} &n_f(p)b(k)[1-n_f(k)][1+b(p)] \\ &+\bar{n}_f(p)b(k)[1-\bar{n}_f(k)][1+b(p)]. \end{aligned} \tag{25}$$

The χ-function terms becomes

$$\begin{aligned} &\left[\chi_{ij}^{\mathrm{e}}(\mathbf{p})+\chi_{ij}^{\gamma}(\mathbf{k})-\chi_{ij}^{\mathrm{e}}(\mathbf{k})-\chi_{ij}^{\gamma}(\mathbf{p})\right]^2 \\ &\longrightarrow\left[\chi^{\mathrm{e}}(p)-\chi^{\gamma}(p)\right]^2+\left[\chi^{\mathrm{e}}(k)-\chi^{\gamma}(k)\right]^2. \end{aligned} \tag{26}$$

After finishing the integration over $\mathrm{d}k$ we can recast (11) into

$$\begin{aligned} &(\chi_{ij},\mathcal{C}\chi_{ij})^{(c)} \qquad (27)\\ &=\frac{\beta e^4\ln\frac{1}{e}}{2^3\pi^5}\int_0^\infty \mathrm{d}pp\left[\chi_{ij}^{\mathrm{e}}(p)-\chi_{ij}^{\gamma}(p)\right]^2 \\ &\times\left\{n_f(p)[1+b(p)]\left(\frac{\pi^2}{8}-0.270\frac{\mu}{T}-0.048\frac{\mu^2}{T^2}\right)\right. \\ &\left.+\bar{n}_f(p)[1+b(p)]\left(\frac{\pi^2}{8}-0.270\frac{\mu}{T}-0.048\frac{\mu^2}{T^2}\right)\right\}. \end{aligned}$$

4 Variational method

As far as shear viscosity is concerned, two species of particles are involved and $\chi(p)$ must take two components:

$$\chi(p)=\begin{pmatrix}\chi^{\mathrm{e}}(p)\\ \chi^{\gamma}(p)\end{pmatrix}. \tag{28}$$

Accordingly the collision operator $\mathcal{C}$ is a 2×2 matrix. The left hand side of the Boltzmann equation, (7), reads

$$\begin{aligned} &(\chi_{ij},S_{ij}) \\ &=-\frac{\beta^2}{\pi^2}\int_0^\infty \mathrm{d}pp^3\left\{b(p)[1+b(p)]\chi^{\gamma}(p)\right. \\ &+n_f(p)[1-n_f(p)]\chi^{\mathrm{e}}(p) \\ &\left.+\bar{n}_f(p)[1-\bar{n}_f(p)]\chi^{\mathrm{e}}(p)\right\}. \end{aligned} \tag{29}$$

Since we have already obtained all the collision terms in the Boltzmann equation, we are going to solve

$$(\chi_{ij},S_{ij})=(\chi_{ij},\mathcal{C}\chi_{ij}) \tag{30}$$

to get the shear viscosity by varying the ansatz χ_{ij} to reach its maximum value. We are not going to argue much about the accuracy of this method in this paper, because Arnold et al. [9] have compared it with the exact results at high temperature but zero chemical potential environment. And we will see that the ansatz in the pure temperature environment is a function only in terms of the momentum and the thermal variables; thereby we can safely use the same ansatz form for small μ.

Before we choose the exact ansatz of χ^{γ} and χ^{e}, we prefer to demonstrate the scheme of this variational calculus. For simplicity all the subscripts and the momentum dependences of each function and operator are dropped, and the Boltzmann equation becomes

$$(\chi,S)=(\chi,\mathcal{C}\chi). \tag{31}$$

Expanding the χ-function in a finite basis set,

$$\chi=\sum_{m=1}^{N}a_m\phi_m=\boldsymbol{a}\cdot\boldsymbol{\phi}, \tag{32}$$

one finds that (31) becomes

$$\sum_m a_m(\phi_m,S)=\sum_{mn}a_m a_n(\phi_m,\mathcal{C}\phi_n). \tag{33}$$

Redefining S and $\mathcal{C}$ in the ϕ_m basis set, one finds

$$\sum_m a_m\tilde{S}_m=\sum_{mn}a_m\tilde{\mathcal{C}}a_n, \tag{34}$$

with $\tilde{S}\equiv(\phi_m,S)$ and $\tilde{\mathcal{C}}=(\phi_m,\mathcal{C}\phi_n)$. It is a trivial exercise to give $\boldsymbol{a}=\tilde{\mathcal{C}}^{-1}\tilde{S}$ and

$$\eta=\frac{2}{15}Q_{\max}=\frac{1}{15}\boldsymbol{a}\cdot\tilde{S}=\frac{1}{15}\tilde{S^{\top}}\tilde{\mathcal{C}}^{-1}\tilde{S}. \tag{35}$$

For the real two-component χ-function one can expand it in the same finite basis set,

$$\chi^{\gamma}(p)=\sum_{m=1}^{N}a_m\phi_m(p),\quad \chi^{\mathrm{e}}(p)=\sum_{m=1}^{N}a_{N+m}\phi_m(p), \tag{36}$$

where $\{a_1,a_2,\ldots,a_{2N}\}$ are the independent variational parameters, and we adopt the one function of the set with the natural ansatz $\phi(p)=p^2$,

$$\chi(p)=\begin{pmatrix}a_1\phi(p)\\ a_2\phi(p)\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\end{pmatrix}p^2. \tag{37}$$

By using this form of ansatz and neglecting the higher orders than μ^2/T^2, one can evaluate

$$\begin{aligned} \tilde{S}&=-\frac{\beta^2}{\pi^2}\int_0^\infty \mathrm{d}pp^5 \\ &\times\begin{pmatrix}b(p)[1+b(p)]\\ n_f(p)[1-n_f(p)]+\bar{n}_f(p)[1-\bar{n}_f(p)]\end{pmatrix} \\ &=-\frac{120\xi(5)T^4}{\pi^2}\begin{pmatrix}1\\ \frac{15}{8}\left(1+0.869\frac{\mu^2}{T^2}\right)\end{pmatrix} \end{aligned} \tag{38}$$

by expanding the fermion distribution function in terms of μ/T and neglecting the higher orders of μ^2/T^2.

The collision term $\tilde{\mathcal{C}}$ can be obtained likewise by combining (17), (23) and (27):

$$\tilde{\mathcal{C}} = \frac{\pi T^5 e^4 \ln e^{-1}}{9} \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{7}{4}\left(1 + 0.738\frac{\mu^2}{T^2}\right) \end{pmatrix} + \frac{9\pi^2}{128}\left(1 + 0.443\frac{\mu^2}{T^2}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]. \quad (39)$$

Inserting (38) and (39) into (35) we obtain the shear viscous coefficient for the QED plasma:

$$\eta_{\text{QED}} = 187.13\frac{T^3}{e^4 \ln e^{-1}}\left(1 + 0.13\frac{\mu^2}{T^2} + \mathcal{O}\left(\frac{\mu^4}{T^4}\right)\right), \quad (40)$$

which recovers the result of [9] at $\mu = 0$ and has a structure similar to that from the relaxation time approximation [14, 20].

5 Discussion and outlook

So far we have obtained the shear viscosity of the QED plasma at finite temperature and density in the leading-log order. The chemical potential modifies the result in the pure temperature case by an additional term which is proportional to $\mu^2 T$, which ensures that the modification factor is irrelevant to the sign of the net charge of the plasma due to the symmetry. In addition, the sign in front of the modification factor is positive, which indicates that the chemical potential increases the shear viscosity of the plasma. Although we obtain this result in the small μ limit, the tendency keeps unchanged in the whole region of $\mu < T$.

In thermal field theory, we can also obtain such a kind of result as (40) by replacing the damping rate by the transport damping rate [21] in the boson-exchange case. The reason for this replacement is clear when one looks into the kinetic theory: the extra q^2 coming from the χ-function in $ee \to ee$ scattering softens the quadratic divergence into a logarithmical one. This extra small q^2 appeared only in the boson-exchange process and is the origin of the extra $\sin^2\frac{\theta}{2}$ in the transport damping rate. Carrington, Defu and Kobes also pointed out [5], that these χ terms can be explained as an infinite series of resummed ladder diagrams. These facts imply that the one-loop calculation with the usual interaction rate is not complete even in the amplitude of order. But the replacement of the transport interaction rate improves the calculation and makes the results reliable.

In a weakly coupled system the fermion thermal mass is of order eT, which is much smaller than the hard momentum scale T and thus can be neglected. Even when we assume that the chemical potential is in the same order of eT, it is still reasonable to omit the mass term in the distribution functions for the reason that if one expands the terms on the exponential of the distribution function in terms of the small fermion mass,

$$\beta(\sqrt{p^2 + m^2} \pm \mu) \approx \beta\left(p + \frac{m^2}{2p} \pm \mu\right), \quad (41)$$

it is easy to check that the second term is of order $e^2 T$. However, the mass effect is quite interesting when one consider the heavy fermion in heavy-ion physics. For example, Aarts and Resco [22] have evaluated the shear viscosity and electric conductivity in gauge theory with massive fermions in the large N_f expansion and found both coefficients to go to zero for large mass.

We have calculated the viscosity of the plasma involving only $2 \to 2$ processes to leading logarithm. But the inelastic scatterings and interference effects might be important if we go beyond the leading-log and obtain the complete leading order contribution. Furthermore, to explain the near-perfect property of QGP, one needs to treat the strong coupling system. In this case we have to use the Kubo formula and calculate the correlation functions of the relevant currents.

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